Numerical Analysis

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10E

Numerical Analysis

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TENTH EDITION

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This edition is dedicated to the memory of
J. Douglas Faires
Doug was a friend, colleague, and coauthor for over 40 years.
He will be sadly missed.

Contents

Preface

About the Text

This text was written for a sequence of courses on the theory and application of numerical approximation techniques. It is designed primarily for junior-level mathematics, science, and engineering majors who have completed at least the first year of the standard college calculus sequence. Familiarity with the fundamentals of matrix algebra and differential equations is useful, but there is sufficient introductory material on these topics so that courses in these subjects are not needed as prerequisites.

Previous editions of *Numerical Analysis* have been used in a wide variety of situations. In some cases, the mathematical analysis underlying the development of approximation techniques was given more emphasis than the methods; in others, the emphasis was reversed. The book has been used as a core reference for beginning graduate-level courses in engineering, mathematics, computer science programs, and in first-year courses in introductory analysis offered at international universities. We have adapted the book to fit these diverse users without compromising our original purpose:

> *To introduce modern approximation techniques; to explain how, why, and when they can be expected to work; and to provide a foundation for further study of numerical analysis and scientific computing.*

The book contains sufficient material for at least a full year of study, but we expect many people will use the text only for a single-term course. In such a single-term course, students learn to identify the types of problems that require numerical techniques for their solution and see examples of the error propagation that can occur when numerical methods are applied. They accurately approximate the solution of problems that cannot be solved exactly and learn typical techniques for estimating error bounds for their approximations. The remainder of the text then serves as a reference for methods that are not considered in the course. Either the full-year or the single-course treatment is consistent with the philosophy of the text.

Virtually every concept in the text is illustrated by example, and this edition contains more than 2500 class-tested exercises ranging from elementary applications of methods and algorithms to generalizations and extensions of the theory. In addition, the exercise sets include numerous applied problems from diverse areas of engineering as well as from the physical, computer, biological, economic, and social sciences. The applications, chosen clearly and concisely, demonstrate how numerical techniques can and often must be applied in real-life situations.

A number of software packages, known as Computer Algebra Systems (CAS), have been developed to produce symbolic mathematical computations. Maple©, Mathematica©, and MATLAB© are predominant among these in the academic environment. Student versions of these software packages are available at reasonable prices for most common computer systems. In addition, Sage, a free open source system, is now available. Information about this system can be found at the site

http://www.sagemath.org

Although there are differences among the packages, both in performance and in price, all can perform standard algebra and calculus operations.

The results in most of our examples and exercises have been generated using problems for which exact values *can* be determined because this better permits the performance of the approximation method to be monitored. In addition, for many numerical techniques, the error analysis requires bounding a higher ordinary or partial derivative of a function, which can be a tedious task and one that is not particularly instructive once the techniques of calculus have been mastered. So having a symbolic computation package available can be very useful in the study of approximation techniques because exact solutions can often be obtained easily using symbolic computation. Derivatives can be quickly obtained symbolically, and a little insight often permits a symbolic computation to aid in the bounding process as well.

Algorithms and Programs

In our first edition, we introduced a feature that at the time was innovative and somewhat controversial. Instead of presenting our approximation techniques in a specific programming language (FORTRAN was dominant at the time), we gave algorithms in a pseudocode that would lead to a well-structured program in a variety of languages. Beginning with the second edition, we listed programs in specific languages in the *Instructor's Manual* for the book, and the number of these languages increased in subsequent editions. We now have the programs coded and available online in most common programming languages and CAS worksheets. All of these are on the companion website for the book (see "Supplements").

For each algorithm, there is a program written in Fortran, Pascal, C, and Java. In addition, we have coded the programs using Maple, Mathematica, and MATLAB. This should ensure that a set of programs is available for most common computing systems.

Every program is illustrated with a sample problem that is closely correlated to the text. This permits the program to be run initially in the language of your choice to see the form of the input and output. The programs can then be modified for other problems by making minor changes. The form of the input and output are, as nearly as possible, the same in each of the programming systems. This permits an instructor using the programs to discuss them generically without regard to the particular programming system an individual student uses.

The programs are designed to run on a minimally configured computer and given in ASCII format to permit flexibility of use. This permits them to be altered using any editor or word processor that creates standard ASCII files. (These are also commonly called "text-only" files.) Extensive README files are included with the program files so that the peculiarities of the various programming systems can be individually addressed. The README files are presented both in ASCII format and as PDF files. As new software is developed, the programs will be updated and placed on the website for the book.

For most of the programming systems, the appropriate software is needed, such as a compiler for Pascal, Fortran, and C, or one of the computer algebra systems (Maple, Mathematica, and MATLAB). The Java implementations are an exception. You need the system to run the programs, but Java can be freely downloaded from various sites. The best way to obtain Java is to use a search engine to search on the name, choose a download site, and follow the instructions for that site.

New for This Edition

The first edition of this book was published more than 35 years ago, in the decade after major advances in numerical techniques were made to reflect the new widespread availability of computer equipment. In our revisions of the book, we have added new techniques in an attempt to keep our treatment current. To continue this trend, we have made a number of significant changes for this edition:

- Some of the examples in the book have been rewritten to better emphasize the problem being solved before the solution is given. Additional steps have been added to some of the examples to explicitly show the computations required for the first steps of iteration processes. This gives readers a way to test and debug programs they have written for problems similar to the examples.
- Chapter exercises have been split into computational, applied, and theoretical to give the instructor more flexibility in assigning homework. In almost all of the computational situations, the exercises have been paired in an odd-even manner. Since the odd problems are answered in the back of the text, if even problems were assigned as homework, students could work the odd problems and check their answers prior to doing the even problem.
- Many new applied exercises have been added to the text.
- Discussion questions have been added after each chapter section primarily for instructor use in online courses.
- The last section of each chapter has been renamed and split into four subsections: Numerical Software, Discussion Questions, Key Concepts, and Chapter Review. Many of these discussion questions point the student to modern areas of research in software development.
- Parts of the text were reorganized to facilitate online instruction.
- Additional PowerPoints have been added to supplement the reading material.
- The bibliographic material has been updated to reflect new editions of books that we reference. New sources have been added that were not previously available.

As always with our revisions, every sentence was examined to determine if it was phrased in a manner that best relates what we are trying to describe.

Supplements

The authors have created a companion website containing the supplementary materials listed below. The website located at

https://sites.google.com/site/numericalanalysis1burden/

is for students and instructors. Some material on the website is for instructor use only. Instructors can access protected materials by contacting the authors for the password.

Some of the supplements can also be obtained at

https://www.cengagebrain.com

by searching the ISBN.

1. *Student Program Examples* that contain Maple, Matlab, and Excel code for student use in solving text problems. This is organized to parallel the text chapter by chapter. Commands in these systems are illustrated. The commands are presented in very short program segments to show how exercises may be solved without extensive programming.

2. *Student Lectures* that contain additional insight to the chapter content. These lectures were written primarily for the online learner but can be useful to students taking the course in a traditional setting.

3. *Student Study Guide* that contains worked-out solutions to many of the problems. The first two chapters of this guide are available on the website for the book in PDF format so that prospective users can tell if they find it sufficiently useful. The entire guide can be obtained only from the publisher by calling Cengage Learning Customer & Sales Support at 1-800-354-9706 or by ordering online at http://www.cengagebrain.com/.

4. *Algorithm Programs* that are complete programs written in Maple, Matlab, Mathematica, C, Pascal, Fortran, and Java for all the algorithms in the text. These programs are intended for students who are more experienced with programming languages.

5. *Instructor PowerPoints* in PDF format for instructor use in both traditional and online courses. Contact authors for password.

6. *Instructor's Manual* that provides answers and solutions to all the exercises in the book. Computation results in the *Instructor's Manual* were regenerated for this edition using the programs on the website to ensure compatibility among the various programming systems. Contact authors for password.

7. *Instructor Sample Tests* for instructor use. Contact authors for password.

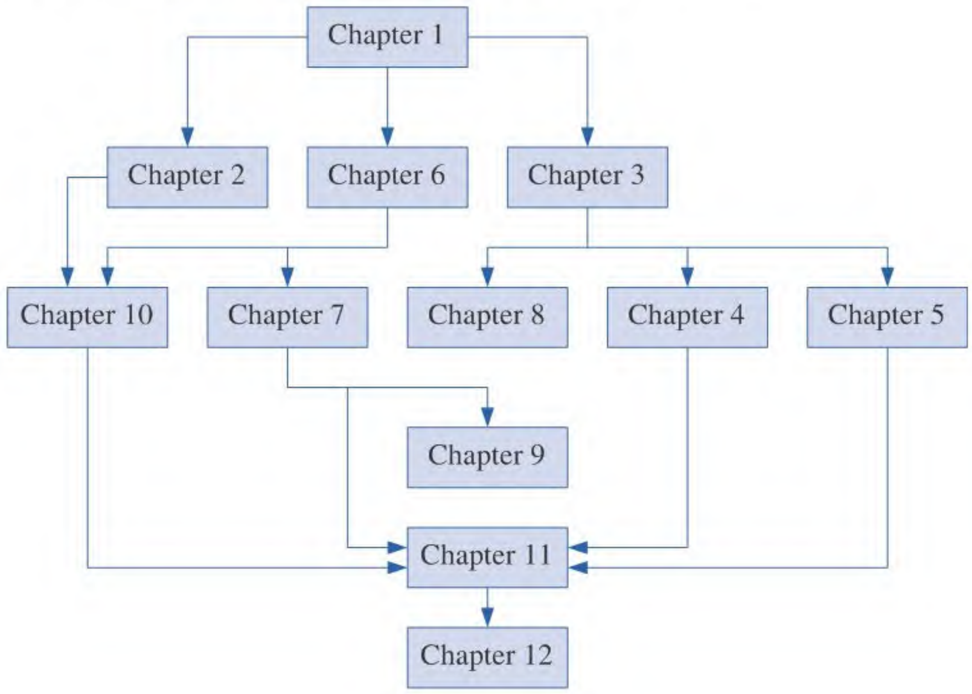
8. *Errata*.

Possible Course Suggestions

Numerical Analysis is designed to allow instructors flexibility in the choice of topics as well as in the level of theoretical rigor and in the emphasis on applications. In line with these aims, we provide detailed references for the results that are not demonstrated in the text and for the applications that are used to indicate the practical importance of the methods. The text references cited are those most likely to be available in college libraries and have been updated to reflect recent editions. We also include quotations from original research papers when we feel this material is accessible to our intended audience. All referenced material has been indexed to the appropriate locations in the text, and Library of Congress call information for reference material has been included to permit easy location if searching for library material.

The following flowchart indicates chapter prerequisites. Most of the possible sequences that can be generated from this chart have been taught by the authors at Youngstown State University.

The material in this edition should permit instructors to prepare an undergraduate course in numerical linear algebra for students who have not previously studied numerical analysis. This could be done by covering Chapters 1, 6, 7, and 9.



Acknowledgments

We have been fortunate to have had many of our students and colleagues give us their impressions of earlier editions of this book, and we take all of these comments very seriously. We have tried to include all the suggestions that complement the philosophy of the book and are extremely grateful to all those that have taken the time to contact us about ways to improve subsequent versions.

We would particularly like to thank the following, whose suggestions we have used in this and previous editions.

Douglas Carter,

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Christopher Harrison,

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Mikhail M. Shvartsman, University of St. Thomas

Dennis C. Smolarski, Santa Clara University

Dale Smith, Comcast

We would like to thank Dr. Barbara T. Faires for her cooperation in providing us with materials that we needed to make this revision possible. Her graciousness during such a difficult time was greatly appreciated.

As has been our practice in past editions of the book, we have used student help at Youngstown State University in preparing the tenth edition. Our able assistant for this edition was Teryn Jones who worked on the Java applets. We would like to thank Edward R. Burden, an Electrical Engineering doctoral student at Ohio State University, who has been checking all the application problems and new material in the text. We also would like to express gratitude to our colleagues on the faculty and administration of Youngstown State University for providing us the opportunity and facilities to complete this project.

We would like to thank some people who have made significant contributions to the history of numerical methods. Herman H. Goldstine has written an excellent book titled *A History of Numerical Analysis from the 16th Through the 19th Century* [Golds]. Another source of excellent historical mathematical knowledge is the MacTutor History of Mathematics archive at the University of St. Andrews in Scotland. It has been created by John J. O'Connor and Edmund F. Robertson and has the Internet address

http://www-gap.dcs.st-and.ac.uk/~history/

An incredible amount of work has gone into creating the material on this site, and we have found the information to be unfailingly accurate. Finally, thanks to all the contributors to Wikipedia who have added their expertise to that site so that others can benefit from their knowledge.

In closing, thanks again to those who have spent the time and effort to contact us over the years. It has been wonderful to hear from so many students and faculty who used our book for their first exposure to the study of numerical methods. We hope this edition continues this exchange and adds to the enjoyment of students studying numerical analysis. If you have any suggestions for improving future editions of the book, we would, as always, be grateful for your comments. We can be contacted most easily by e-mail at the addresses listed below.

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CHAPTER

1

Mathematical Preliminaries and Error Analysis

Introduction

In beginning chemistry courses, we see the *ideal gas law*,

$$PV = NRT,$$

which relates the pressure P, volume V, temperature T, and number of moles N of an "ideal" gas. In this equation, R is a constant that depends on the measurement system.

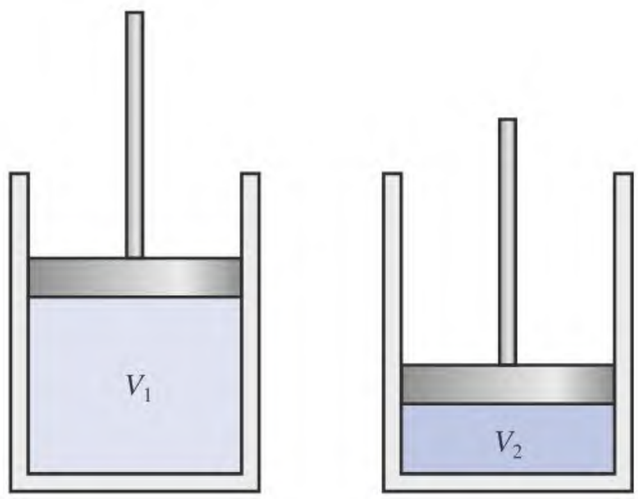
Suppose two experiments are conducted to test this law, using the same gas in each case. In the first experiment,

$$P = 1.00 \text{ atm}, \qquad V = 0.100 \text{ m}^3,$$
$$N = 0.00420 \text{ mol}, \qquad R = 0.08206.$$

The ideal gas law predicts the temperature of the gas to be

$$T = \frac{PV}{NR} = \frac{(1.00)(0.100)}{(0.00420)(0.08206)} = 290.15 \text{ K} = 17^\circ\text{C}.$$

When we measure the temperature of the gas, however, we find that the true temperature is 15°C.



We then repeat the experiment using the same values of R and N but increase the pressure by a factor of two and reduce the volume by the same factor. The product PV remains the same, so the predicted temperature is still 17°C. But now we find that the actual temperature of the gas is 19°C.

Clearly, the ideal gas law is suspect, but before concluding that the law is invalid in this situation, we should examine the data to see whether the error could be attributed to the experimental results. If so, we might be able to determine how much more accurate our experimental results would need to be to ensure that an error of this magnitude does not occur.

Analysis of the error involved in calculations is an important topic in numerical analysis and is introduced in Section 1.2. This particular application is considered in Exercise 26 of that section.

This chapter contains a short review of those topics from single-variable calculus that will be needed in later chapters. A solid knowledge of calculus is essential for an understanding of the analysis of numerical techniques, and more thorough review might be needed for those who have been away from this subject for a while. In addition there is an introduction to convergence, error analysis, the machine representation of numbers, and some techniques for categorizing and minimizing computational error.

1.1 Review of Calculus

Limits and Continuity

The concepts of *limit* and *continuity* of a function are fundamental to the study of calculus and form the basis for the analysis of numerical techniques.

Definition 1.1 A function f defined on a set X of real numbers has the **limit** L at x_0, written

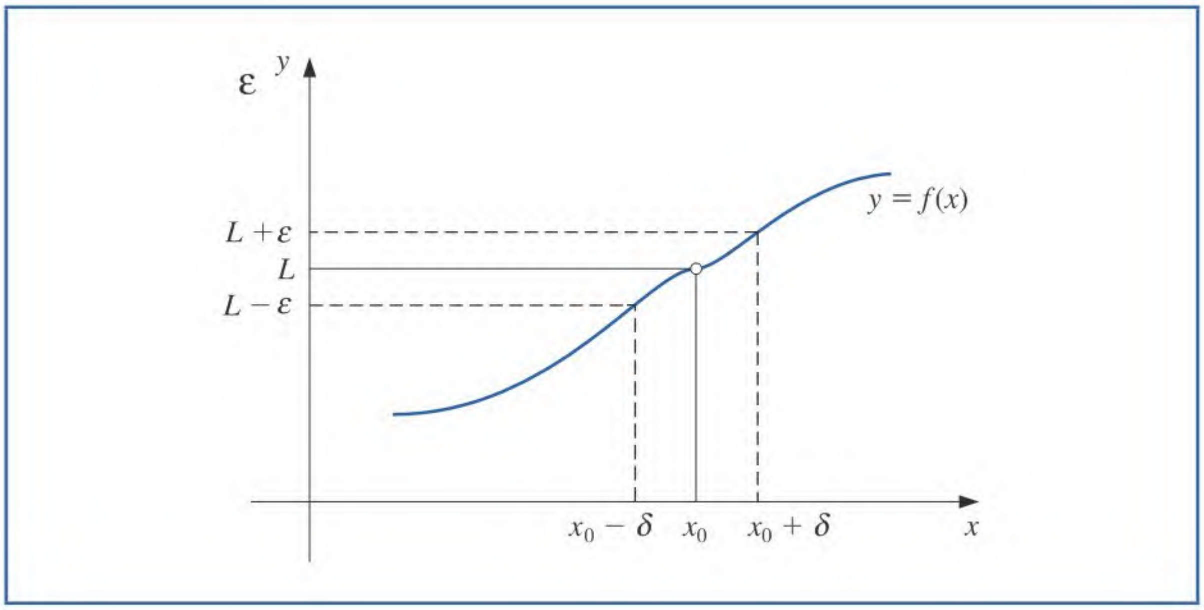
$$\lim_{x \to x_0} f(x) = L,$$

if, given any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon, \quad \text{whenever} \quad x \in X \quad \text{and} \quad 0 < |x - x_0| < \delta.$$

(See Figure 1.1.) ■

Figure 1.1



Definition 1.2 Let f be a function defined on a set X of real numbers and $x_0 \in X$. Then f is **continuous** at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

The function f is **continuous on the set** X if it is continuous at each number in X. ■

The basic concepts of calculus and its applications were developed in the late 17th and early 18th centuries, but the mathematically precise concepts of limits and continuity were not described until the time of Augustin Louis Cauchy (1789–1857), Heinrich Eduard Heine (1821–1881), and Karl Weierstrass (1815 –1897) in the latter portion of the 19th century.

The set of all functions that are continuous on the set X is denoted $C(X)$. When X is an interval of the real line, the parentheses in this notation are omitted. For example, the set of all functions continuous on the closed interval $[a, b]$ is denoted $C[a, b]$. The symbol $\mathbb{R}$ denotes the set of all real numbers, which also has the interval notation $(-\infty, \infty)$. So the set of all functions that are continuous at every real number is denoted by $C(\mathbb{R})$ or by $C(-\infty, \infty)$.

The *limit of a sequence* of real or complex numbers is defined in a similar manner.

Definition 1.3 Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers. This sequence has the **limit** x (**converges to** x) if, for any $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that $|x_n - x| < \varepsilon$ whenever $n > N(\varepsilon)$. The notation

$$\lim_{n \to \infty} x_n = x, \quad \text{or} \quad x_n \to x \quad \text{as} \quad n \to \infty,$$

means that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x. ■

Theorem 1.4 If f is a function defined on a set X of real numbers and $x_0 \in X$, then the following statements are equivalent:

a. f is continuous at x_0;

b. If $\{x_n\}_{n=1}^{\infty}$ is any sequence in X converging to x_0, then $\lim_{n \to \infty} f(x_n) = f(x_0)$.

■

The functions we will consider when discussing numerical methods will be assumed to be continuous because this is a minimal requirement for predictable behavior. Functions that are not continuous can skip over points of interest, which can cause difficulties in attempts to approximate a solution to a problem.

Differentiability

More sophisticated assumptions about a function generally lead to better approximation results. For example, a function with a smooth graph will normally behave more predictably than one with numerous jagged features. The smoothness condition relies on the concept of the derivative.

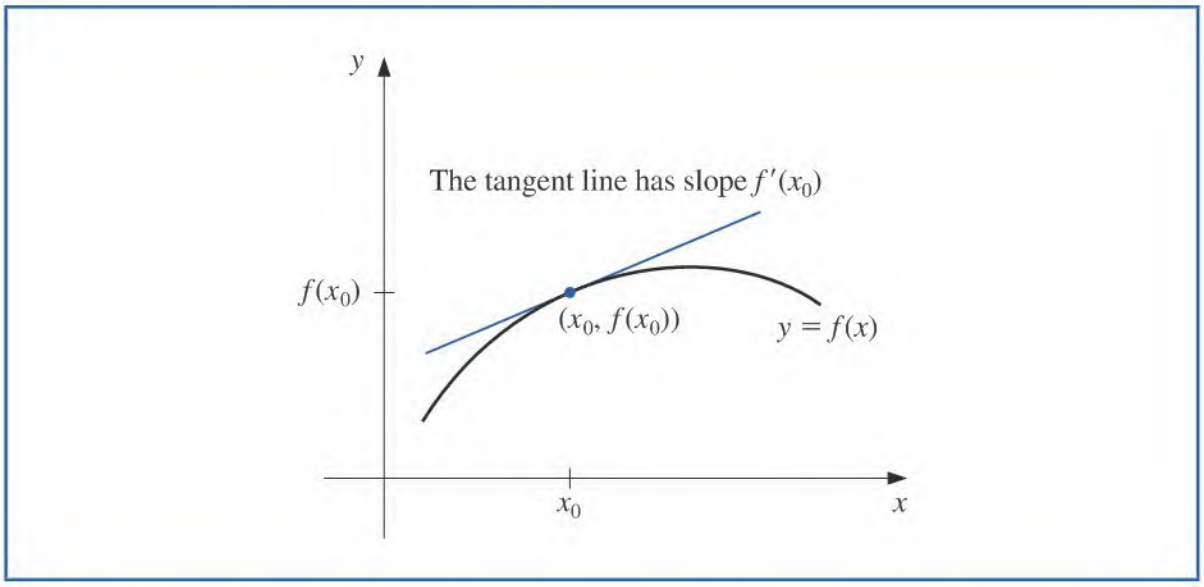
Definition 1.5 Let f be a function defined in an open interval containing x_0. The function f is **differentiable** at x_0 if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The number $f'(x_0)$ is called the **derivative** of f at x_0. A function that has a derivative at each number in a set X is **differentiable on** X. ■

The derivative of f at x_0 is the slope of the tangent line to the graph of f at $(x_0, f(x_0))$, as shown in Figure 1.2.

Figure 1.2



Theorem 1.6 If the function f is differentiable at x_0, then f is continuous at x_0. ■

The theorem attributed to Michel Rolle (1652–1719) appeared in 1691 in a little-known treatise titled *Méthode pour résoundre les égalites*. Rolle originally criticized the calculus that was developed by Isaac Newton and Gottfried Leibniz but later became one of its proponents.

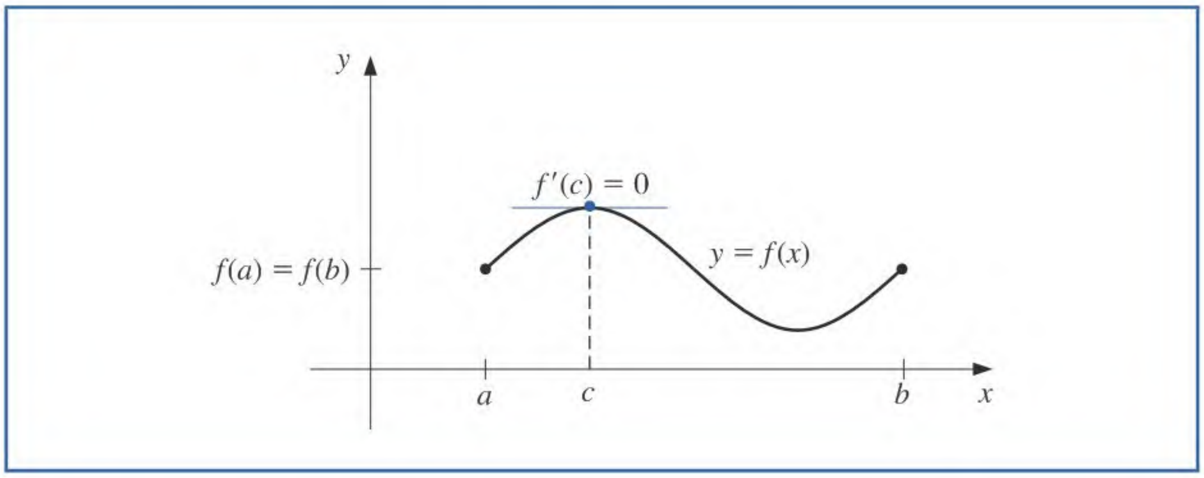
The next theorems are of fundamental importance in deriving methods for error estimation. The proofs of these theorems and the other unreferenced results in this section can be found in any standard calculus text.

The set of all functions that have n continuous derivatives on X is denoted $C^n(X)$, and the set of functions that have derivatives of all orders on X is denoted $C^\infty(X)$. Polynomial, rational, trigonometric, exponential, and logarithmic functions are in $C^\infty(X)$, where X consists of all numbers for which the functions are defined. When X is an interval of the real line, we will again omit the parentheses in this notation.

Theorem 1.7 **(Rolle's Theorem)**

Suppose $f \in C[a, b]$ and f is differentiable on (a, b). If $f(a) = f(b)$, then a number c in (a, b) exists with $f'(c) = 0$. (See Figure 1.3.) ■

Figure 1.3

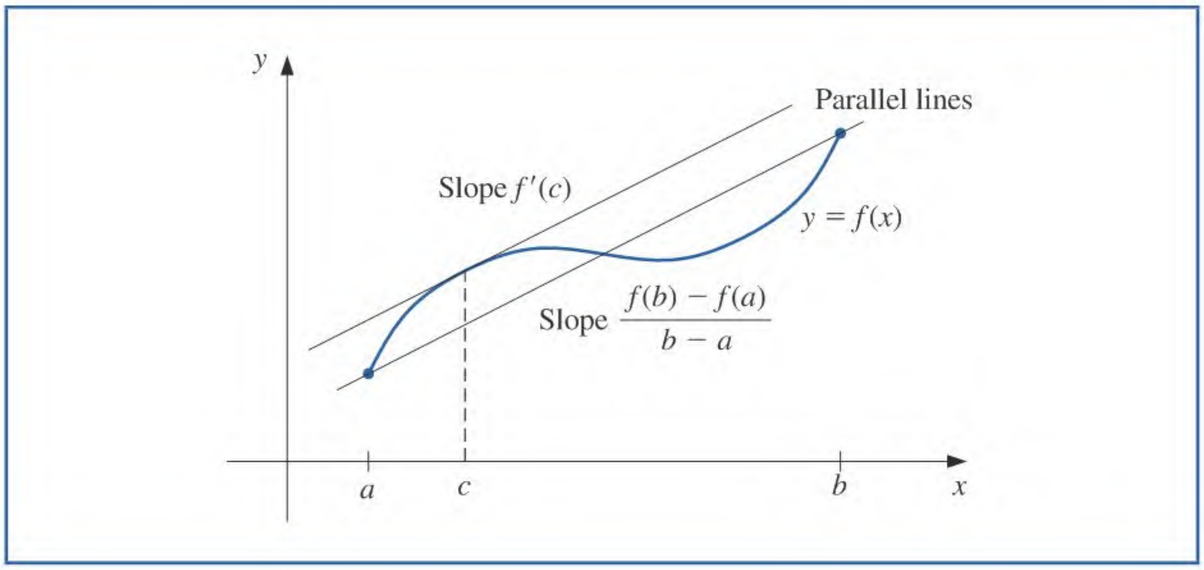


Theorem 1.8 **(Mean Value Theorem)**

If $f \in C[a, b]$ and f is differentiable on (a, b), then a number c in (a, b) exists with (See Figure 1.4.)

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ ■

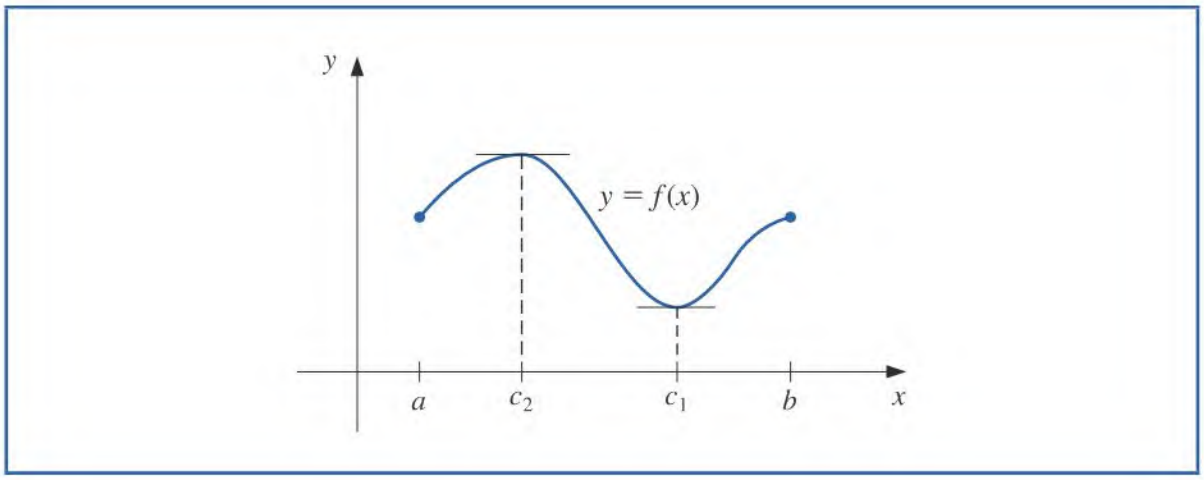
Figure 1.4



Theorem 1.9 **(Extreme Value Theorem)**

If $f \in C[a, b]$, then $c_1, c_2 \in [a, b]$ exist with $f(c_1) \leq f(x) \leq f(c_2)$, for all $x \in [a, b]$. In addition, if f is differentiable on (a, b), then the numbers c_1 and c_2 occur either at the endpoints of $[a, b]$ or where f' is zero. (See Figure 1.5.) ■

Figure 1.5



Example 1 Find the absolute minimum and absolute maximum values of

$$f(x) = 2 - e^x + 2x$$

on the intervals **(a)** [0, 1], and **(b)** [1, 2].

Solution We begin by differentiating $f(x)$ to obtain

$$f'(x) = -e^x + 2.$$

$f'(x) = 0$ when $-e^x + 2 = 0$ or, equivalently, when $e^x = 2$. Taking the natural logarithm of both sides of the equation gives

$$\ln(e^x) = \ln(2) \text{ or } x = \ln(2) \approx 0.69314718056$$

(a) When the interval is [0, 1], the absolute extrema must occur at $f(0)$, $f(\ln(2))$, or $f(1)$. Evaluating, we have

$$f(0) = 2 - e^0 + 2(0) = 1$$

$$f(\ln(2)) = 2 - e^{\ln(2)} + 2\ln(2) = 2\ln(2) \approx 1.38629436112$$

$$f(1) = 2 - e + 2(1) = 4 - e \approx 1.28171817154.$$

Thus, the absolute minimum of $f(x)$ on [0, 1] is $f(0) = 1$ and the absolute maximum is $f(\ln(2)) = 2\ln(2)$.

(b) When the interval is [1, 2], we know that $f'(x) \neq 0$ so the absolute extrema occur at $f(1)$ and $f(2)$. Thus, $f(2) = 2 - e^2 + 2(2) = 6 - e^2 \approx -1.3890560983$. The absolute minimum on [1, 2] is $6 - e^2$ and the absolute maximum is 1.

We note that

$$\max_{0 \le x \le 2} |f(x)| = |6 - e^2| \approx 1.3890560983. \quad ■$$

The following theorem is not generally presented in a basic calculus course but is derived by applying Rolle's Theorem successively to f, f', ... , and, finally, to $f^{(n-1)}$. This result is considered in Exercise 26.

Theorem 1.10 **(Generalized Rolle's Theorem)**

Suppose $f \in C[a, b]$ is n times differentiable on (a, b). If $f(x) = 0$ at the $n + 1$ distinct numbers $a \le x_0 < x_1 < \ldots < x_n \le b$, then a number c in (x_0, x_n) and hence in (a, b) exists with $f^{(n)}(c) = 0$. ■

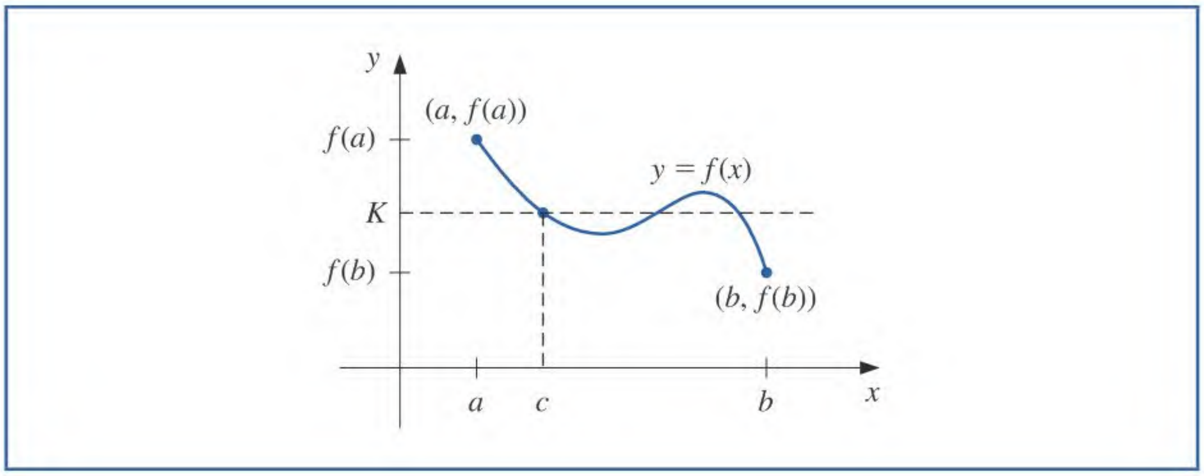
We will also make frequent use of the Intermediate Value Theorem. Although its statement seems reasonable, its proof is beyond the scope of the usual calculus course. It can, however, be found in most analysis texts (see, for example, [Fu], p. 67).

Theorem 1.11 **(Intermediate Value Theorem)**

If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) for which $f(c) = K$. ■

Figure 1.6 shows one choice for the number that is guaranteed by the Intermediate Value Theorem. In this example, there are two other possibilities.

Figure 1.6



Example 2 Show that $x^5 - 2x^3 + 3x^2 - 1 = 0$ has a solution in the interval [0, 1].

Solution Consider the function defined by $f(x) = x^5 - 2x^3 + 3x^2 - 1$. The function f is continuous on [0, 1]. In addition,

$$f(0) = -1 < 0 \quad \text{and} \quad 0 < 1 = f(1).$$

Hence, the Intermediate Value Theorem implies that a number c exists, with $0 < c < 1$, for which $c^5 - 2c^3 + 3c^2 - 1 = 0$. ■

As seen in Example 2, the Intermediate Value Theorem is used to determine when solutions to certain problems exist. It does not, however, give an efficient means for finding these solutions. This topic is considered in Chapter 2.

Integration

The other basic concept of calculus that will be used extensively is the Riemann integral.

Definition 1.12 The **Riemann integral** of the function f on the interval $[a, b]$ is the following limit, provided it exists:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(z_i)\,\Delta x_i,$$

where the numbers $x_0, x_1, \ldots, x_n$ satisfy $a = x_0 \le x_1 \le \cdots \le x_n = b$, where $\Delta x_i = x_i - x_{i-1}$, for each $i = 1, 2, \ldots, n$, and z_i is arbitrarily chosen in the interval $[x_{i-1}, x_i]$. ■

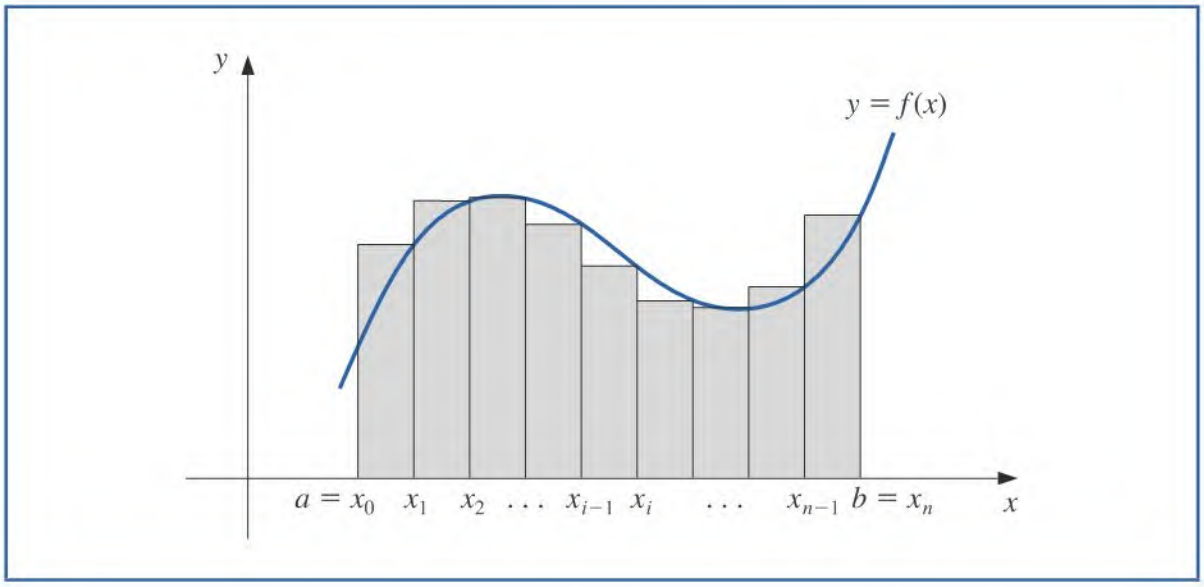
George Fredrich Berhard Riemann (1826–1866) made many of the important discoveries classifying the functions that have integrals. He also did fundamental work in geometry and complex function theory and is regarded as one of the profound mathematicians of the 19th century.

A function f that is continuous on an interval $[a, b]$ is also Riemann integrable on $[a, b]$. This permits us to choose, for computational convenience, the points x_i to be equally spaced in $[a, b]$ and, for each $i = 1, 2, \ldots, n$, to choose $z_i = x_i$. In this case,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i),$$

where the numbers shown in Figure 1.7 as x_i are $x_i = a + i(b - a)/n$.

Figure 1.7



Two other results will be needed in our study of numerical analysis. The first is a generalization of the usual Mean Value Theorem for Integrals.

Theorem 1.13 **(Weighted Mean Value Theorem for Integrals)**

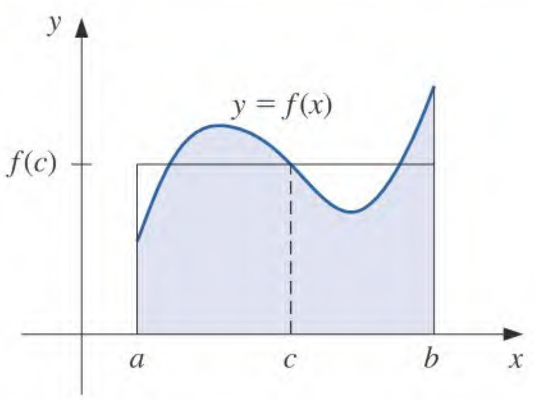
Suppose $f \in C[a, b]$, the Riemann integral of g exists on $[a, b]$, and $g(x)$ does not change sign on $[a, b]$. Then there exists a number c in (a, b) with

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$ ■

When $g(x) \equiv 1$, Theorem 1.13 is the usual Mean Value Theorem for Integrals. It gives the **average value** of the function f over the interval $[a, b]$ as (See Figure 1.8.)

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Figure 1.8



The proof of Theorem 1.13 is not generally given in a basic calculus course but can be found in most analysis texts (see, for example, [Fu], p. 162).

Taylor Polynomials and Series

The final theorem in this review from calculus describes the Taylor polynomials. These polynomials are used extensively in numerical analysis.

Theorem 1.14 **(Taylor's Theorem)**

Suppose $f \in C^n[a, b]$, $f^{(n+1)}$ exists on $[a, b]$, and $x_0 \in [a, b]$. For every $x \in [a, b]$, there exists a number $\xi(x)$ between x_0 and x with

Brook Taylor (1685–1731) described this series in 1715 in the paper *Methodus incrementorum directa et inversa*. Special cases of the result and likely the result itself had been previously known to Isaac Newton, James Gregory, and others.

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}.$$

■

Colin Maclaurin (1698–1746) is best known as the defender of the calculus of Newton when it came under bitter attack by Irish philosopher Bishop George Berkeley.

Maclaurin did not discover the series that bears his name; it was known to century mathematicians before he was born. However, he did devise a method for solving a system of linear equations that is known as Cramer's rule, which Cramer did not publish until 1750.

Here $P_n(x)$ is called the **nth Taylor polynomial** for f about x_0, and $R_n(x)$ is called the **remainder term** (or **truncation error**) associated with $P_n(x)$. Since the number $\xi(x)$ in the truncation error $R_n(x)$ depends on the value of x at which the polynomial $P_n(x)$ is being evaluated, it is a function of the variable x. However, we should not expect to be able to explicitly determine the function $\xi(x)$. Taylor's Theorem simply ensures that such a function exists and that its value lies between x and x_0. In fact, one of the common problems in numerical methods is to try to determine a realistic bound for the value of $f^{(n+1)}(\xi(x))$ when x is in some specified interval.

The infinite series obtained by taking the limit of $P_n(x)$ as $n \to \infty$ is called the **Taylor series** for f about x_0. In the case $x_0 = 0$, the Taylor polynomial is often called a **Maclaurin polynomial**, and the Taylor series is often called a **Maclaurin series**.

The term **truncation error** in the Taylor polynomial refers to the error involved in using a truncated, or finite, summation to approximate the sum of an infinite series.

Example 3 Let $f(x) = \cos x$ and $x_0 = 0$. Determine

(a) the second Taylor polynomial for f about x_0; and

(b) the third Taylor polynomial for f about x_0.

Solution Since $f \in C^\infty(\mathbb{R})$, Taylor's Theorem can be applied for any $n \geq 0$. Also,

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad \text{and} \quad f^{(4)}(x) = \cos x,$$

so

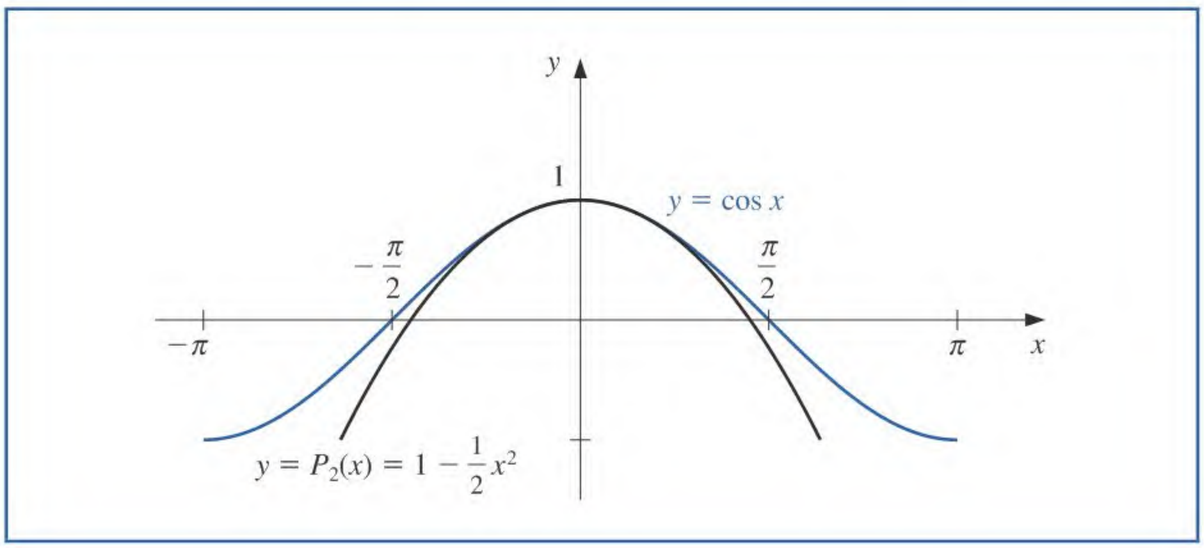
$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad \text{and} \quad f'''(0) = 0.$$

(a) For $n = 2$ and $x_0 = 0$, we have

$$\begin{aligned}\cos x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(\xi(x))}{3!}x^3 \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 \sin \xi(x),\end{aligned}$$

where $\xi(x)$ is some (generally unknown) number between 0 and x. (See Figure 1.9.)

Figure 1.9



When $x = 0.01$, this becomes

$$\cos 0.01 = 1 - \frac{1}{2}(0.01)^2 + \frac{1}{6}(0.01)^3 \sin\xi(0.01) = 0.99995 + \frac{10^{-6}}{6}\sin\xi(0.01).$$

The approximation to $\cos 0.01$ given by the Taylor polynomial is therefore 0.99995. The truncation error, or remainder term, associated with this approximation is

$$\frac{10^{-6}}{6}\sin\xi(0.01) = 0.1\overline{6} \times 10^{-6}\sin\xi(0.01),$$

where the bar over the 6 in $0.1\overline{6}$ is used to indicate that this digit repeats indefinitely. Although we have no way of determining $\sin\xi(0.01)$, we know that all values of the sine lie in the interval $[-1, 1]$, so the error occurring if we use the approximation 0.99995 for the value of $\cos 0.01$ is bounded by

$$|\cos(0.01) - 0.99995| = 0.1\overline{6} \times 10^{-6}|\sin\xi(0.01)| \le 0.1\overline{6} \times 10^{-6}.$$

Hence, the approximation 0.99995 matches at least the first five digits of $\cos 0.01$, and

$$\begin{aligned} 0.9999483 < 0.99995 - 1.\overline{6} \times 10^{-6} &\le \cos 0.01 \\ &\le 0.99995 + 1.\overline{6} \times 10^{-6} < 0.9999517. \end{aligned}$$

The error bound is much larger than the actual error. This is due in part to the poor bound we used for $|\sin\xi(x)|$. It is shown in Exercise 27 that for all values of x, we have $|\sin x| \le |x|$. Since $0 \le \xi < 0.01$, we could have used the fact that $|\sin\xi(x)| \le 0.01$ in the error formula, producing the bound $0.1\overline{6} \times 10^{-8}$.

(b) Since $f'''(0) = 0$, the third Taylor polynomial with remainder term about $x_0 = 0$ is

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\cos\tilde{\xi}(x),$$

where $0 < \tilde{\xi}(x) < 0.01$. The approximating polynomial remains the same, and the approximation is still 0.99995, but we now have much better accuracy assurance. Since $|\cos\tilde{\xi}(x)| \le 1$ for all x, we have

$$\left|\frac{1}{24}x^4\cos\tilde{\xi}(x)\right| \le \frac{1}{24}(0.01)^4(1) \approx 4.2 \times 10^{-10}.$$

So,

$$|\cos 0.01 - 0.99995| \le 4.2 \times 10^{-10},$$

and

$$\begin{aligned} 0.99994999958 &= 0.99995 - 4.2 \times 10^{-10} \\ &\le \cos 0.01 \le 0.99995 + 4.2 \times 10^{-10} = 0.99995000042. \end{aligned}$$ ■

Example 3 illustrates the two objectives of numerical analysis:

(i) Find an approximation to the solution of a given problem.

(ii) Determine a bound for the accuracy of the approximation.

The Taylor polynomials in both parts provide the same answer to (i), but the third Taylor polynomial gave a much better answer to (ii) than the second Taylor polynomial.

We can also use the Taylor polynomials to give us approximations to integrals.

Illustration We can use the third Taylor polynomial and its remainder term found in Example 3 to approximate $\int_0^{0.1} \cos x\, dx$. We have

$$\begin{aligned}\int_0^{0.1} \cos x\, dx &= \int_0^{0.1} \left(1 - \frac{1}{2}x^2\right) dx + \frac{1}{24}\int_0^{0.1} x^4 \cos \tilde{\xi}(x)\, dx \\ &= \left[x - \frac{1}{6}x^3\right]_0^{0.1} + \frac{1}{24}\int_0^{0.1} x^4 \cos \tilde{\xi}(x)\, dx \\ &= 0.1 - \frac{1}{6}(0.1)^3 + \frac{1}{24}\int_0^{0.1} x^4 \cos \tilde{\xi}(x)\, dx.\end{aligned}$$

Therefore,

$$\int_0^{0.1} \cos x\, dx \approx 0.1 - \frac{1}{6}(0.1)^3 = 0.0998\overline{3}.$$

A bound for the error in this approximation is determined from the integral of the Taylor remainder term and the fact that $|\cos \tilde{\xi}(x)| \leq 1$ for all x:

$$\begin{aligned}\frac{1}{24}\left|\int_0^{0.1} x^4 \cos \tilde{\xi}(x)\, dx\right| &\leq \frac{1}{24}\int_0^{0.1} x^4 |\cos \tilde{\xi}(x)|\, dx \\ &\leq \frac{1}{24}\int_0^{0.1} x^4\, dx = \frac{(0.1)^5}{120} = 8.\overline{3} \times 10^{-8}.\end{aligned}$$

The true value of this integral is

$$\int_0^{0.1} \cos x\, dx = \sin x\Big]_0^{0.1} = \sin 0.1 \approx 0.099833416647,$$

so the actual error for this approximation is 8.3314×10^{-8}, which is within the error bound. ■

EXERCISE SET 1.1

1. Show that the following equations have at least one solution in the given intervals.
   a. $x \cos x - 2x^2 + 3x - 1 = 0$, $[0.2, 0.3]$ and $[1.2, 1.3]$
   b. $(x-2)^2 - \ln x = 0$, $[1, 2]$ and $[e, 4]$
   c. $2x \cos(2x) - (x-2)^2 = 0$, $[2, 3]$ and $[3, 4]$
   d. $x - (\ln x)^x = 0$, $[4, 5]$
2. Show that the following equations have at least one solution in the given intervals.
   a. $\sqrt{x} - \cos x = 0$, $[0, 1]$
   b. $e^x - x^2 + 3x - 2 = 0$, $[0, 1]$
   c. $-3\tan(2x) + x = 0$, $[0, 1]$
   d. $\ln x - x^2 + \frac{5}{2}x - 1 = 0$, $[\frac{1}{2}, 1]$
3. Find intervals containing solutions to the following equations.
   a. $x - 2^{-x} = 0$
   b. $2x\cos(2x) - (x+1)^2 = 0$
   c. $3x - e^x = 0$
   d. $x + 1 - 2\sin(\pi x) = 0$

4. Find intervals containing solutions to the following equations.

a. $x - 3^{-x} = 0$

b. $4x^2 - e^x = 0$

c. $x^3 - 2x^2 - 4x + 2 = 0$

d. $x^3 + 4.001x^2 + 4.002x + 1.101 = 0$

5. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

a. $f(x) = (2 - e^x + 2x)/3, \quad [0, 1]$

b. $f(x) = (4x - 3)/(x^2 - 2x), \quad [0.5, 1]$

c. $f(x) = 2x\cos(2x) - (x - 2)^2, \quad [2, 4]$

d. $f(x) = 1 + e^{-\cos(x-1)}, \quad [1, 2]$

6. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

a. $f(x) = 2x/(x^2 + 1), \quad [0, 2]$

b. $f(x) = x^2\sqrt{(4 - x)}, \quad [0, 4]$

c. $f(x) = x^3 - 4x + 2, \quad [1, 2]$

d. $f(x) = x\sqrt{(3 - x^2)}, \quad [0, 1]$

7. Show that $f'(x)$ is 0 at least once in the given intervals.

a. $f(x) = 1 - e^x + (e - 1)\sin((\pi/2)x), \quad [0, 1]$

b. $f(x) = (x - 1)\tan x + x\sin \pi x, \quad [0, 1]$

c. $f(x) = x\sin \pi x - (x - 2)\ln x, \quad [1, 2]$

d. $f(x) = (x - 2)\sin x \ln(x + 2), \quad [-1, 3]$

8. Suppose $f \in C[a, b]$ and $f'(x)$ exists on (a, b). Show that if $f'(x) \ne 0$ for all x in (a, b), then there can exist at most one number p in $[a, b]$ with $f(p) = 0$.

9. Let $f(x) = x^3$.

a. Find the second Taylor polynomial $P_2(x)$ about $x_0 = 0$.

b. Find $R_2(0.5)$ and the actual error in using $P_2(0.5)$ to approximate $f(0.5)$.

c. Repeat part (a) using $x_0 = 1$.

d. Repeat part (b) using the polynomial from part (c).

10. Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x + 1}$ about $x_0 = 0$. Approximate $\sqrt{0.5}$, $\sqrt{0.75}$, $\sqrt{1.25}$, and $\sqrt{1.5}$ using $P_3(x)$ and find the actual errors.

11. Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos x$ about $x_0 = 0$.

a. Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_2(0.5)|$ using the error formula and compare it to the actual error.

b. Find a bound for the error $|f(x) - P_2(x)|$ in using $P_2(x)$ to approximate $f(x)$ on the interval $[0, 1]$.

c. Approximate $\int_0^1 f(x)\,dx$ using $\int_0^1 P_2(x)\,dx$.

d. Find an upper bound for the error in (c) using $\int_0^1 |R_2(x)\,dx|$ and compare the bound to the actual error.

12. Repeat Exercise 11 using $x_0 = \pi/6$.

13. Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = (x - 1)\ln x$ about $x_0 = 1$.

a. Use $P_3(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_3(0.5)|$ using the error formula and compare it to the actual error.

b. Find a bound for the error $|f(x) - P_3(x)|$ in using $P_3(x)$ to approximate $f(x)$ on the interval $[0.5, 1.5]$.

c. Approximate $\int_{0.5}^{1.5} f(x)\,dx$ using $\int_{0.5}^{1.5} P_3(x)\,dx$.

d. Find an upper bound for the error in (c) using $\int_{0.5}^{1.5} |R_3(x)\,dx|$ and compare the bound to the actual error.

14. Let $f(x) = 2x\cos(2x) - (x - 2)^2$ and $x_0 = 0$.

a. Find the third Taylor polynomial $P_3(x)$ and use it to approximate $f(0.4)$.